A priori estimates for conformal mappings on complex plane with parallel slits

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Abstract

We study the properties of a conformal mapping $z(k)$ from the plane without vertical slits $\Gamma_n = [u_n - ih_n, u_n + ih_n], n \in \mathbb{Z}$ and $h = (h_n)_{n\in\mathbb{Z}} \in \ell^2$, onto the complex plane without horizontal slits $\gamma_n \subset \mathbb{R}, n \in \mathbb{Z}$, with the asymptotics $z(iv) = iv + o(1),\ v \to \infty$. Here $u_{n+1} - u_n \geqslant 1, n \in \mathbb{Z}$. Introduce the sequences $l = (|\gamma_n|)_{n\in\mathbb{Z}}$. We obtain a priori two-sided estimates for $\|h\|_{p,\omega}, \|l\|_{p,\omega}$, where the norm $\|h\|_{p,\omega}^p = \sum \omega_n |h_n|^p, 1 \leqslant p \leqslant 2$ with any weight $\omega_n \geqslant 1, n \in \mathbb{Z}$. Moreover, we determine other estimates.

1 Introduction and main results

Consider a conformal mapping $z : K_+(h) \to \mathbb{C}_+$ with asymptotics $z(iv) = iv(1 + o(1))$ as $v \to \infty$, where $z(k), k = u + iv \in K(h)$. Here the domain $K_+(h) = \mathbb{C}_+ \cap K(h)$ for some sequence $h = (h_n)_{n\in\mathbb{Z}} \in \ell^\infty, h_n \geqslant 0$ and the domain $K(h)$ is given by

$$K(h) = \mathbb{C} \setminus \cup_{n\in\mathbb{Z}}\Gamma_n, \quad \Gamma_n = [u_n - ih_n, u_n + ih_n], \quad u_* = \inf_n(u_{n+1} - u_n) \geqslant 0, \tag{1.1}$$

where $u_n, n \in \mathbb{Z}$ is strongly increasing sequence of real numbers such that $u_n \to \pm\infty$ as $n \to \pm\infty$. We fix the sequence $u_n, n \in \mathbb{Z}$ and consider the conformal mapping for various $h \in \ell^\infty$. The difference of any two such mappings equals a real constant. Thus the imaginary part $y(k) = \operatorname{Im} z(k)$ is unique. We call such mapping $z(k)$ the comb mapping. Define the inverse mapping $k(\cdot) : \mathbb{C}_+ \to K_+(h)$. It is clear that $k(z), z = x + iy \in \mathbb{C}_+$ has the continuous extension into $\overline{\mathbb{C}_+}$. We define "gaps" γ_n, "bands" σ_n and the "spectrum" σ of the comb mapping by:

$$\gamma_n = (z_n^-, z_n^+) = (z(u_n - 0), z(u_n + 0)), \qquad \sigma_n = [z_{n-1}^+, z_n^-], \qquad \sigma = \cup_{n\in\mathbb{Z}}\sigma_n.$$

The function $u(z) = \operatorname{Re} k(z)$ is strongly increasing on each band σ_n and $u(z) = u_n$ for all $z \in [z_n^-, z_n^+],\ n \in \mathbb{Z}$; the function $v(z) = \operatorname{Im} k(z)$ equals zero on each band σ_n and is strongly

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convex on each gap $\gamma_n \neq \emptyset$ and has the maximum at some point z_n given by $v(z_n) = h_n$. If the gap is empty we set $z_n = z_n^{\pm}$. The function $z(\cdot)$ has an analytic extension (by the symmetry) from the domain $K_+(h)$ onto the domain $K(h)$ and $z(\cdot) : K(h) \to z(K(h)) = \mathcal{Z} = \mathbb{C} \setminus \cup \overline{\gamma}_n$ is a conformal mapping. These and others properties of the comb mappings it is possible to find in the papers of Levin [Le].

For any $p \geqslant 1$ and the weight $\omega = (\omega_n)_{n\in\mathbb{Z}}$, where $\omega_n \geqslant 1$, we introduce the real spaces

$$\ell^p_\omega = \{f = (f_n)_{n\in\mathbb{Z}} : \quad \|f\|_{p,\omega} < \infty\}, \quad \|f\|^p_{p,\omega} = \sum_{n\in\mathbb{Z}} \omega_n f_n^p < \infty.$$

If the weight $\omega_n = (2u_n)^{2m}, m \in \mathbb{R}$ for all $n \in \mathbb{Z}$, then we will write ℓ^p_m with the norm $\|\cdot\|_{p,m}$. If the weight $\omega_n = 1$ for all $n \in \mathbb{Z}$, then we will write $\ell^p_0 = \ell^p$ with the norm $\|\cdot\|_p$ and $\|\cdot\| = \|\cdot\|_2$. For each $h = (h_n)_{n\in\mathbb{Z}}$ we introduce the sequences

$$l = (l_n)_{n\in\mathbb{Z}}, \quad l_n = |\gamma_n|, \quad J = (J_n)_{n\in\mathbb{Z}}, \quad J_n = |A_n|^{\frac{1}{2}} \geqslant 0, \quad A_n = \frac{2}{\pi}\int_{\gamma_n} v(z)dz \geqslant 0.$$

For the defocussing cubic non-linear Schrödinger equation (a completely integrable infinite dimensional Hamiltonian system), $k(z)$ is a quasi-momentum and A_n is an action variables (see [K6]). We formulate the first main result about the estimates in terms of $\|\cdot\|_p$.

Theorem 1.1. *Let $u_* = \inf_n(u_{n+1} - u_n) > 0$. Then the following estimates hold:*

$$\|h\|_p \leqslant 2\|l\|_p(1 + \alpha_p \ \|l\|_p^p), \quad 1 \leqslant p \leqslant 2, \quad \alpha_p = (2^{p+2}(2+\pi)/u_*)^p/\pi, \tag{1.2}$$

$$\|h\|_p \leqslant \frac{2}{\pi} C_p^2 \|l\|_q \Big(1 + \Big[\frac{2C_p}{\pi u_*}\Big]^{\frac{2}{p-1}} \|l\|_q^{\frac{2}{p-1}}\Big), \quad C_p = \Big(\frac{\pi^2}{2}\Big)^{1/p}, \quad p \geqslant 2, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{1.3}$$

$$\frac{\|l\|_p}{2} \leqslant \|J\|_p \leqslant \frac{2}{\sqrt{\pi}}\|l\|_p(1 + \alpha_p\|l\|_p^p)^{1/2}, \tag{1.4}$$

$$\frac{\sqrt{\pi}}{2}\|J\|_p \leqslant \|h\|_p \leqslant 4\|J\|_p(1 + \alpha_p 2^p \|J\|_p^p). \tag{1.5}$$

Estimates (1.2)-(1.5) are new for $p \in [1, 2)$. Korotyaev [K1] obtained the two-sided estimates for the case $p = 2$ (see Theorem 2.4), for example

$$\frac{1}{2}\|l\| \leqslant \|h\| \leqslant \pi\|l\|\Big(1 + \frac{2}{u_*^2}\|l\|^2\Big), \quad if \quad u_* > 0. \tag{1.6}$$

Introduce the effective masses $\mu_n^{\pm}$ for the ends $z_n^- < z_n^+$ by

$$z(k) - z_n^{\pm} = \frac{(k - u_n)^2}{2\mu_n^{\pm}} + O((k - u_n)^3) \qquad \text{as} \quad z \to z_n^{\pm}. \tag{1.7}$$

If $|\gamma_n| = 0$, then we set $\mu_n^{\pm} = 0$. Define the sequence $\mu^{\pm} = (\mu_n^{\pm})_{n\in\mathbb{Z}}$. We formulate the second result.

Theorem 1.2. *Let $h \in \ell^p_\omega, p \in [1,2]$ and let $u_* > 0$. Then the following estimates hold:*

$$\|h\|_\infty \leqslant \min\{2\pi\|\mu^\pm\|_\infty,\ \|J\|_{p,\omega},\ 2\pi^{-1/p}\|l\|_{p,\omega}(1+\alpha_p\|l\|^p_{p,\omega})^{1/q}\}, \tag{1.8}$$

$$\|l\|_{p,\omega} \leqslant 2\|h\|_{p,\omega} \leqslant \xi^9\|l\|_{p,\omega}, \qquad \xi = \exp(\|h\|_\infty/u_*), \tag{1.9}$$

$$\|l\|_{p,\omega} \leqslant 2\|J\|_{p,\omega} \leqslant \xi^5 2\|l\|_{p,\omega}, \tag{1.10}$$

$$\frac{\sqrt{\pi}}{2}\|J\|_{p,\omega} \leqslant \|h\|_{p,\omega} \leqslant \xi^5\sqrt{\frac{\pi}{2}}\|J\|_{p,\omega}, \tag{1.11}$$

$$\|l\|_{p,\omega} \leqslant 2\|\mu^\pm\|_{p,\omega} \leqslant \xi^{18}\|l\|_{p,\omega}. \tag{1.12}$$

Estimates (1.8)-(1.12) are new. Korotyaev obtained the two-sided estimates for the space $\ell^2_m, m \geqslant 0$ for the even case $h_{-n} = h_n, n \in \mathbb{Z}$ ([K2]-[K4]) and for the space ℓ^2_1 without symmetry ([K1], ([K6])). In all these estimate the factor $\xi = \exp(\|h\|_\infty/u_*)$ is absent.

Proposition 1.3. *i) Estimate* (1.2) *at* $p = 1$ *is sharp.*
ii) Estimate (2.10) *is sharp.*
iii) If the estimate $\|h\| \leqslant C\|l\|(1+\|l\|^p),\ \ p > 0$ *is true, then* $p \geqslant 1$.

Recall that for a compact subset $\Omega \subset \mathbb{C}$ the analytic capacity is given by

$$\mathcal{C} = \mathcal{C}(\Omega) = \sup\Big[|f'(\infty)| : f \textit{ is analytic in } \mathbb{C}\setminus\Omega; \quad |f(k)| \leqslant 1,\ k \in \mathbb{C}\setminus\Omega\Big], \tag{1.13}$$

where $f'(\infty) = \lim_{|k|\to\infty} k(f(k) - f(\infty))$. We will use the well known Theorem (see [Iv], [Po])

Theorem *(Ivanov-Pommerenke). Let $E \subset \mathbb{R}$ be compact. Then the analytic capacity $\mathcal{C}(E) = |E|/4$, where $|E|$ is the Lebesgue measure (the length) of the set E. Moreover, the Ahlfors function f_E (the unique function, which gives* sup *in the definition of the analytic capacity) has the following form:*

$$f_E(z) = \frac{\exp\left(\frac12\phi_E(z)\right) - 1}{\exp\left(\frac12\phi_E(z)\right) + 1}, \qquad \phi_E(z) = \int_E \frac{dt}{z-t}; \quad z \in \mathbb{C}\setminus E. \tag{1.14}$$

We will use the following simple remark: Let $S_1, S_2, \dots, S_N$ be disjoint continua in the plane $\mathbb{C}$; $D = \mathbb{C}\setminus\cup_{n=1}^N S_n$. Introduce the class $\Sigma'(D)$ of the conformal mapping w from the domain D onto $\mathbb{C}$ with the following asymptotics: $w(k) = k + [Q(w) + o(1)]/k,\ k \to \infty$. If $\Omega \subset \mathbb{C}$ is compact; $D = \mathbb{C}\setminus\Omega,\ g \in \Sigma'(D)$, then $\mathcal{C}(\Omega) = \mathcal{C}(\mathbb{C}\setminus g(D))$. It follows immediately from the definition of the analytic capacity.

Let $\Phi_+ \subset \ell^\infty$ be the subset of finite sequences of non negative numbers. Then, using the Ivanov-Pommerenke Theorem and the last remark we obtain

$$\|l(h)\|_1 = \mathcal{C}(\Gamma(h)), \quad where \quad h \in \Phi_+,\ \Gamma(h) = \cup[u_n - ih_n, u_n + ih_n];$$

Proposition 1.4. *Let $h \in \ell^\infty$ and let $u_* \geqslant 0$. Then*

$$\|h\|_\infty^2 \leqslant 2Q_0 = \frac{1}{\pi}\int_{\mathbb{R}} v(x)\, dx, \tag{1.15}$$

$$\pi Q_0 \leqslant \|h\|_p \|l\|_q, \quad p \geqslant 1, \tag{1.16}$$

$$I_D \leqslant (\frac{2}{\pi})^{2/p} \|h\|_p^{2/q} \|l\|_p^{2/p}, \quad 1 \leqslant p \leqslant 2, \tag{1.17}$$

$$\pi Q_0 \leqslant \|h\|_\infty \|l\|_1 \leqslant \frac{2}{\pi}\|l\|_1^2, \tag{1.18}$$

$$\|h\|_\infty \leqslant \frac{2}{\pi}\|l\|_1, \quad \|l\|_1 \leqslant 2\|h\|_1. \tag{1.19}$$

Below we will sometimes write $\gamma_n(h), z(k,h), ..,$ instead of $\gamma_n, z(k), ..,$ when several sequences $h \in \ell^\infty$ are being dealt with. Define the Dirichlet integral

$$I_D(h) = \frac{1}{\pi}\iint_{\mathbb{C}} |z'(k,h) - 1|^2\, du\, dv = \frac{1}{\pi}\iint_{\mathbb{C}} |k'(z,h) - 1|^2\, dx\, dy, \;\; k = u + iv, \;\; z = x + iy.$$

The last identity holds since the Dirichlet integral is invariant under the conformal mappings.

Now we estimate the Dirichlet integral $I_D(h)$ (or $Q_0(h) = \frac{1}{\pi}\int_{\mathbb{R}} v(x)dx$) for the case $u_* \geqslant 0$, using the following geometric construction. For the vector $h \in \ell^\infty, h_n \to 0$ as $n \to \infty$, we introduce the sequence $\widetilde{h} = \widetilde{h}(h)$ by: if $h = 0$, then $\widetilde{h} = 0$,

if $h \neq 0$, then we take an integer n_1 such that $\widetilde{h}_{n_1} = h_{n_1} = \max_{n\in\mathbb{Z}} h_n > 0$; assume that we define the numbers $h_{n_1}, h_{n_2}, \ldots, h_{n_k}$, then we take n_{k+1} such that

$$\widetilde{h}_{n_{k+1}} = h_{n_{k+1}} = \max_{n\in B} h_n > 0, \quad B = \{n \in \mathbb{Z} : |u_n - u_{n_l}| > h_{n_l}, 1 \leqslant l \leqslant k\}. \tag{1.20}$$

Moreover, we let $\widetilde{h}_n = 0$, if $n \notin \{n_k, k \in \mathbb{Z}\}$. Now we formulate the following results

Theorem 1.5. *Let $h \in \ell^\infty, h_n \to 0$ as $|n| \to \infty$; and let $\widetilde{h} = \widetilde{h}(h)$. Then the following estimates hold:*

$$\frac{1}{\pi^2}\|\widetilde{h}\|_2^2 \leqslant Q_0(h) = \frac{I_D(h)}{2} \leqslant \frac{2\sqrt{2}}{\pi}\|\widetilde{h}\|_2^2. \tag{1.21}$$

We give the geometry interpretation of the estimates from all these theorems. Define the square differential in the domain $D = K(h) \cup \{\infty\}$ on the Riemann sphere, which is considered as the Riemann surface with hyperbolic boundary components by:

$$w = (k'(z,h) - 1)^2 dz^2 = (z'(k,h) - 1)^2 dk^2.$$

Then w is the analytic square differential on D (in particular, analytic at any boundary point in terms of a corresponding uniformizing parameter). In the present paper the metric w is important to get the needed estimates. Moreover, these estimates have the following geometry interpretation. The invariant length L_n of the cut γ_n has the following form

$$L_n = 2\int_{z_n^-}^{z_n^+} |k'(x) - 1| dx = 2\int_{z_n^-}^{z_n^+} \sqrt{v'(x)^2 + 1} dx.$$

Then using (2.1) we obtain $2h_n \leqslant L_n \leqslant 2(h_n + l_n) \leqslant 6h_n$. Moreover, the invariant area S of the Riemann surface D has the form

$$S = \iint_{\mathbb{C}} |k'(z) - 1|^2 dxdy = 2\pi Q_0 = \int_{\mathbb{R}} v(t)dt.$$

Using Theorem 1.1-1.5, we estimate S in terms of $(h_n)_1^N$ or $(L_n)_1^N$.

Levin [Le] proved the existence of the mapping for a very general case. First two-sided estimates for $\|h\|_{2,1}$ and $Q_2 = \frac{1}{\pi}\int_{\mathbb{R}} t^2 v(t)dt$ were obtained in [MO2] only for the case $u_n = \pi n, n \in \mathbb{Z}$. Note that these estimates are overstated since the Bernstein inequality was used. Garnet and Trubowitz [GT] also obtained some estimates, using the different arguments. The authors of the present paper [KK1] obtained estimates of the various parameters, the identities (2.2). First two-sided ewstimates (very rough) for $\|l\|_{2,1}$ and Q_2 were obtained in [KK2]. Identities and various sharp estimates (in terms of gap lengths and effective masses) were obtained by Korotyaev [K1]-[K6].

The estimates for conformal mapping were used to study the inverse problem for the Schrödinger operator with a periodic potential [KK2], [K5-7], for the periodic weighted operator [K8] and for the periodic Zaharov-Shabat systems [K6].

Note that the comb mappings are used in various fields of mathematics. We enumerate the more important directions:
1) the conformal mapping theory, 2) the Löwner equation and the quadratic differentials, 3) the electrostatic problems on the plane, 4) analytic capacity, 5) the spectral theory of the operators with periodic coefficients, 6) inverse problems for the Hill operator and the Dirac operator, 7) KDV equation and NLS equation with periodic initial value problem.

Finally, we shall briefly describe our motivation to derive the present results. Consider the electrostatic field in the domain $K(h) = \mathbb{C} \setminus \cup_{n\in\mathbb{Z}}\Gamma_n$, where $\Gamma_n = [u_n - ih_n, u_n + ih_n], n \in \mathbb{Z}$, is the system of neutral conductors, for some $h \in \ell^p_\omega$ and $u_{n+1} - u_n \geqslant 1,\ n \in \mathbb{Z}$. In other words, we embed the system of neutral conductors $\Gamma_n, n \in \mathbb{Z}$, in the external homogeneous electrostatic field $E_0 = (0, -1) \in \mathbb{R}^2$ on the plane. Then on each conductor there exists the induced charge, positive $e_n > 0$ on the lower half of the conductor Γ_n and negative $(-e_n) < 0$ on the upper half of the conductor Γ_n, since their sum equals zero. As a result we have new perturbed electrostatic field $\mathcal{E} \in \mathbb{R}^2$. It is well known that $\mathcal{E} = \overline{iz'(k,h)} = -\nabla y(k,h),\ k = u + iv \in K(h),\ \ z = x + iy$. Recall that $z(k,h)$ is the conformal mapping from $K(h)$ onto the domain $\mathcal{Z} = \mathbb{C} \setminus \cup\gamma_n$. The function $y(k,h)$ is called the potential of the electrostatic field in $K(h)$. The density of the charge on the conductor has the form $\rho_e(k) = |y'_u(k,h)|/4\pi,\ k \in \Gamma_n$ (see [LS]). Thus we obtain the induced charge e_n on the upper half of the conductor $\Gamma_n^+ = \Gamma_n \cap \mathbb{C}_+$ by:

$$e_n = \frac{1}{4\pi}\int_{\Gamma_n^+} x'_v(k)dv = \frac{1}{4\pi}|\gamma_n|.$$

Introduce the bipolar moment d_n of the conductor Γ_n with the charge density $\rho_e(k)$ by $d_n = \frac{1}{4\pi}\int_{\Gamma_n} v x_v(k)dv \geqslant 0$. We transform this value into the form

$$d_n = \frac{1}{2\pi}\int_{\gamma_n} v(x)dx = \frac{A_n}{4}.$$

In the paper [KK3] we study inverse problems for the charge mapping $h \to e(h) = (e_n(h))_{n\in\mathbb{Z}}$ and the bipolar moment mapping $h \to J(h)$ acting in $\ell^p_\omega, p \in [1,2]$. In order to solve the inverse problems we need a priori estimates from Theorem 1.1 and 1.2.

We now describe the plan of the paper. In Section 2 we shall obtain some preliminaries results and "local basic estimates" in Theorem 2.4. In Section 3 we shall prove the main theorems. Moreover, we consider some examples, which describe our estimates.

2 Preliminaries

We recall needed results. Below we will use very often the following simple estimate

$$l_n \leqslant 2h_n, \quad \text{all} \quad n \in \mathbb{Z}, \tag{2.1}$$

(see e.g. [MO1], [KK1]). Hence if $h \in \ell^p_\omega$, then $l(h) \in \ell^p_\omega$.

For each $h \in \ell^\infty$ the following estimates and identities hold

$$\frac{1}{4}\|l\|^2 \leqslant 2Q_0 = I_D = \sum A_n = \|J\|^2 \leqslant \frac{2}{\pi}\sum_{n\in\mathbb{Z}} h_n l_n, \tag{2.2}$$

$$\max\Big\{\frac{l_n^2}{4}, \frac{l_n h_n}{\pi}\Big\} \leqslant A_n = \frac{2}{\pi}\int_{\gamma_n} v(x)dx \leqslant \frac{2l_n h_n}{\pi}, \tag{2.3}$$

see [KK1]. These show that functional $Q_0 = \frac{1}{\pi}\int_{\mathbb{R}} v(x)\,dx$ is bounded for $h \in \ell^2$. Define the effective masses ν_n in the plane $K(h)$ for the end of the slit $[u_n + ih_n, u_n - ih_n]$ by

$$k(z) - (u_n + ih_n) = \frac{(z - z_n)^2}{2i\nu_n} + O((z - z_n)^3), \qquad z \to z_n. \tag{2.4}$$

Thus we obtain $\nu_n = 1/|k''(z_n)|$, if $h_n > 0$ and we set $\nu_n = 0$ if $l_n = 0$.

Below we will use the Lindelöf principle (see [J]), which is formulated in the form, convenient for us (see [KK1]):

Let $h,\ \widetilde{h} \in \ell^\infty$; and let $\widetilde{h}_n \leqslant h_n$ for all $n \in \mathbb{Z}$. Then the following estimates hold:

$$y(k, \widetilde{h}) \geqslant y(k, h), \quad k \in K_+(h), \tag{2.5}$$

$$Q_0(\widetilde{h}) \leqslant Q_0(h) \ \ \textit{and if}\ \ Q_0(\widetilde{h}) = Q_0(h), \ \ \textit{then}\ \widetilde{h} = h, \tag{2.6}$$

$$l_m(\widetilde{h}) \geqslant l_m(h). \tag{2.7}$$

We show the possibility of this principle in the following Lemma.

Lemma 2.1. *For each $h \in \ell^\infty$ the estimate* (1.15) *and the following estimate hold:*

$$\nu_n \leqslant h_n, \qquad \textit{all} \quad n \in \mathbb{Z}. \tag{2.8}$$

Proof. We apply estimate (2.5) to h and to the new sequence: $\widetilde{h}_m = h_n$ if $m = n$ and $\widetilde{h}_n = 0$ if $m \neq n$. It is clear that $z(k, \widetilde{h}) = \sqrt{(k-u_n)^2 + h_n^2}$ (the principal value). Then

$$y(k,h) \leqslant \operatorname{Im}(\sqrt{(k-u_n)^2+h_n^2}), \ \ k \in K_+(h).$$

Then asymptotics (2.4) of the function $z(k,h)$ as $k \to u_n + ih_n$ yields (2.8). In order to prove (1.15) we use (2.6) since $Q_0(\widetilde{h}) = h_m^2/2$. ■

We recall estimates from [K4].

Theorem 2.2. *Let $h \in \ell^\infty$. Then for any $n \in \mathbb{Z}$ the following estimate holds:*

$$2h_n^2 \leqslant \pi \max\Big\{1, \frac{h_n}{r}\Big\} \iint_{D_n(r)} |z'(k) - 1|^2 du dv, \qquad D_n(r) = (u_n, u_n + r) \times (-h_n, h_n). \quad (2.9)$$

If in addition, $\inf_n(u_{n+1} - u_n) = u_* > 0$, *then*

$$\frac{\pi}{4} I_D \leqslant \|h\|^2 \leqslant \frac{\pi^2}{2} \max\Big\{1, \frac{I_D^{1/2}}{u_*}\Big\} I_D, \quad (2.10)$$

$$\frac{\|l\|}{2} \leqslant \|J\| \leqslant \sqrt{2}\|l\|(1 + \frac{\sqrt{2}}{u_*}\|l\|). \quad (2.11)$$

Introduce the domain $S_r = \{z \in \mathbb{C} : \ |\operatorname{Re} z| < r\}, r > 0$. In order to prove Theorem 2.4 we need the following result about the simple mapping.

Lemma 2.3. *The function $f(k) = \sqrt{k^2 + h^2}$, $k \in \mathbb{C} \setminus [-ih, ih], h > 0$ is the conformal mapping from $\mathbb{C} \setminus [-ih, ih]$ onto $\mathbb{C} \setminus [-h, h]$ and $S_r \setminus [-h, h] \subset f(S_r \setminus [-ih, ih])$ for any $r > 0$.*

Proof. Consider the image of the half-line $k = r + iv, v > 0$. We have the equations

$$x^2 + y^2 = \xi \equiv r^2 + h^2 - v^2, \qquad xy = rv. \quad (2.12)$$

The second identity in (2.12) yields $x > 0$ since $y > 0$. Then $x^4 - \xi x^2 - r^2v^2 = 0$, and enough to check the following inequality $x^2 = \frac{1}{2}(\xi + \sqrt{\xi^2 + 4r^2v^2}) > r^2$. The last estimate follows from the simple relations

$$(r^2 + h^2 - v^2)^2 + 4r^2v^2 > (r^2 + v^2 - h^2)^2, \qquad 4r^2v^2 > 4r^2(v^2 - h^2). \quad ■$$

We prove the local estimates for the small slits. Recall $S_r = \{z \in \mathbb{C} : \ |\operatorname{Re} z| < r\}, r > 0$.

Theorem 2.4. *Let $h \in \ell^\infty$. Assume that $(u_n - r, u_n + r) \subset (u_{n-1}, u_{n+1})$ and $h_n \leqslant \frac{r}{2}$, for some $n \in \mathbb{Z}$ and $r > 0$. Then*

$$|h_n - |\mu_n^\pm|| \leqslant \frac{2+\pi}{r} |\mu_n^\pm| \sqrt{I_n}, \quad I_n = \frac{1}{\pi} \iint_{u_n + S_r} |z'(k) - 1|^2 du dv, \quad (2.13)$$

$$0 \leqslant h_n - \nu_n \leqslant 2\frac{2+\pi}{r} h_n \sqrt{I_n}, \quad (2.14)$$

$$0 \leqslant h_n - \frac{l_n}{2} \leqslant \frac{2+\pi}{r} h_n \sqrt{I_n}. \quad (2.15)$$

Proof. Define the functions $f(k) = \sqrt{k^2 + h_n^2}, \ \ k \in S_r \setminus [-ih_n, ih_n], g = f^{-1}$ and $F(w) = z(u_n + g(w), h), w = p + iq$, where the variable $w \in G_1 = f((S_r \setminus [-ih_n, ih_n]))$. The function F is real for real w, then F is analytic in the domain $G = G_1 \cup [-h_n, h_n]$ and Lemma 2.3 yields $S_r \subset G$. Let now $|w_1| = \frac{r}{2}$ and $B_r = \{z : |z| < r\}$. Then the following estimates hold

$$\sqrt{\pi}\frac{r}{2}|F'(w_1) - 1| \leqslant (\iint_{B_r} |F'(w) - 1|^2 dpdq)^{1/2} \leqslant \tag{2.16}$$

$$\leqslant \Big(\iint_{B_r} |(F(w) - g(w))'|^2 dpdq\Big)^{1/2} + \Big(\iint_{B_r} |g'(w) - 1|^2 dpdq\Big)^{1/2}.$$

The invariance of the Dirichlet integral with respect to the conformal mapping gives

$$\iint_{B_r} |(F(w) - g(w))'|^2 \, dpdq = \iint_{g(B_r)} |z'(k) - 1|^2 dudv \leqslant \iint_{S_r + u_n} |z'(k) - 1|^2 dudv = \pi I_n. \tag{2.17}$$

Moreover, the identity $2Q_0 = I_D$ implies

$$\frac{1}{\pi}\iint_{B_r} |g'(w) - 1|^2 dpdq \leqslant \frac{1}{\pi}\iint_{\mathbb{C}} |g'(w) - 1|^2 dpdq = \frac{2}{\pi}\int_{-h_n}^{h_n} \sqrt{h_n^2 - x^2} \, dx = h_n^2. \tag{2.18}$$

Then (2.16)-(2.18) for $|w_1| = \frac{r}{2}$ yields $|F'(w_1) - 1| \leqslant \frac{2}{r}(\sqrt{I_n} + h_n)$, and (2.9) gives

$$h_n^2 \leqslant \frac{\pi^2}{4}\Big(\frac{1}{\pi}\iint_{S_r + u_n} |z'(k) - 1|^2 dudv\Big) \leqslant \frac{\pi^2}{4} I_n.$$

Then for $|w_1| = r/2$ we have

$$|F'(w_1) - 1| \leqslant \frac{2}{r}(1 + \frac{\pi}{2})\sqrt{I_n} = \frac{2 + \pi}{r}\sqrt{I_n}, \tag{2.19}$$

and the maximum principle yields the needed estimates for $|w_1| \leqslant r/2$.

We prove (2.13) for μ_n^+. The definition of $\mu_n^\pm$ (see (1.7)) implies

$$F'(h_n) = \lim_{x \searrow h_n} z'(u_n + g(x)) \cdot g'(x) = \lim_{x \searrow h_n} \frac{g(x)}{\mu_n^+} \cdot \frac{x}{g(x)} = \frac{h_n}{\mu_n^+}.$$

The substitution of the last identity into (2.19) gives (2.13). The proof for μ_n^- is similar.

We show (2.14). The definition of ν_n (see (2.4)) yields

$$(z(k) - z_n)^2 = 2i\nu_n(k - u_n - ih_n)(1 + o(1)) \ \ as \ \ k \to u_n + ih_n,$$

$$g(w) - ih_n = -\frac{i}{2h_n}(w - z_n)^2(1 + o(1)) \ \ as \ w \to u_n.$$

Then we have $F'(0) = \sqrt{\frac{\nu_n}{h_n}}$ and the substitution of the last identity into (2.19) shows $\Big|\sqrt{\frac{\nu_n}{h_n}} - 1\Big| \leqslant \frac{2+\pi}{r}\sqrt{I_n}$, which gives (2.14), since by (2.3), $\nu_n \leqslant h_n$. Estimate (2.19) yiels

$$0 \leqslant 2h_n - l_n = \int_{-h_n}^{h_n} (1 - F'(x))dx \leqslant \frac{2h_n}{r}(2 + \pi)\sqrt{I_n},$$

which implies (2.15). ■

We prove the estimates in terms of the norm of the space $\ell^p, p \geqslant 1$.

Proof of Theorem 1.4. Estimate (2.2) and the Hölder inequality yield (1.16). Using (1.15), $\pi Q_0 \leqslant \sum h_n l_n$ (see (2.2)) and the Hölder inequality we obtain

$$(\pi/2) I_D \leqslant \|h\|_\infty^{1-\frac{p}{q}} \sum_{n\in\mathbb{Z}} l_n\, h_n^{p/q} \leqslant (I_D)^{(1-\frac{p}{q})/2} \|l\|_p \|h\|_p^{\frac{p}{q}},$$

$$(\pi/2) I_D^{\frac{(1+\frac{p}{q})}{2}} \leqslant \|l\|_p \|h\|_p^{\frac{p}{q}}, \qquad \text{and} \qquad I_D \leqslant \Big(\frac{2}{\pi}\Big)^{\frac{2}{p}} \|l\|_p^{\frac{2}{p}} \|h\|_p^{\frac{2}{q}}.$$

Estimate (1.16) at $q=1$ implies the first one in (1.18). The last result and (1.15) yield the first inequality in (1.19) and then the second one in (1.18). The second estimate in (1.19) follows from $l_n \leqslant 2h_n,\ n \in \mathbb{Z}$ (see (2.1)). We have proved (1.15) in Lemma 2.1. ■

3 Proof of the mains theorems

Proof of Theorem 1.1. Let $1 \leqslant p \leqslant 2$ and $r = \frac{u_*}{2}$. Estimate (2.9) implies

$$h_n^2 \leqslant \frac{\pi^2}{2} \max\{1, \frac{h_n}{r}\} I_n, \quad I_n = \frac{1}{\pi} \iint_{D_n} |z'(k) - 1|^2 du dv, \quad D_n = \{k : |\operatorname{Re}(k - u_n)| < \frac{u_*}{2}\}. \quad (3.1)$$

Hence

$$h_n \leqslant \frac{\pi^2}{u_*} I_n, \quad if \;\; h_n > \frac{u_*}{4}, \;\; and \;\; h_n \leqslant \frac{\pi}{u_*}\sqrt{I_n}, \quad if \;\; h_n \leqslant \frac{u_*}{4}. \quad (3.2)$$

Moreover, (2.15) yields

$$h_n \leqslant \frac{l_n}{2} + 2h_n \frac{2+\pi}{u_*}\sqrt{I_n}, \qquad if \quad h_n < \frac{u_*}{4}, \quad (3.3)$$

and then

$$h_n \leqslant 2\pi \frac{2+\pi}{u_*} I_n, \qquad if \quad h_n < \frac{u_*}{4},\ l_n \leqslant h_n, \quad (3.4)$$

since $h_n \leqslant \frac{\pi}{2}\sqrt{I_n}$. Hence

$$if \quad l_n \leqslant h_n, \;\; then \;\; h_n \leqslant 2\pi \frac{2+\pi}{u_*} I_n = C_1 I_n, \qquad C_1 = 2\pi \frac{2+\pi}{u_*}.$$

The last inequality and (1.15) yield

$$\|h\|_p \leqslant (\sum_{h_n < l_n} h_n^p)^{1/p} + (\sum_{l_n \leqslant h_n} h_n h_+^{p-1})^{1/p} \leqslant \|l\|_p + C_1^{\frac{1}{p}} I_D^{\frac{p+1}{2p}}. \quad (3.5)$$

If we assume that $C_1^{\frac{1}{p}} I_D^{\frac{p+1}{2p}} \leqslant \|l\|_p$, then we obtain $\|h\|_p \leqslant 2\|l\|_p$.

Conversely, let $\|l\|_p \leqslant C_1^{\frac{1}{p}} I_D^{\frac{p+1}{2p}}$. Then (3.5), (1.17) implies

$$\|h\|_p \leqslant 2C_1^{\frac{1}{p}} \Big[(2/\pi)^{2/p} \|h\|_p^{2/q} \, \|l\|_p^{2/p} \Big]^{\frac{p+1}{2p}}.$$

Recall that the constant $\alpha_p = (2^{p+2}(2+\pi)/u_*)^p/\pi$. Hence

$$\|h\|_p^{1/p^2} \leqslant 2C_1^{\frac{1}{p}} \Big[(2/\pi)\|l\|_p \Big]^{\frac{p+1}{p^2}}, \qquad \text{and} \qquad \|h\|_p \leqslant 2^{p^2} C_1^p (2/\pi)^{p+1} \|l\|_p^{1+p},$$

which yields (1.2). Let $p \geqslant 2$. Using inequality (**??**), (1.15) we obtain

$$\|h\|_p \leqslant (\sum h_+^{p-2} h_n^2)^{1/p} \leqslant C_p b^{1/p} I_D^{1/2}, \qquad b = b(h_+), \quad h_+ = \|h\|_\infty. \tag{3.6}$$

Consider the case $b \leqslant 1$. Then (3.6), (1.16) imply

$$\|h\|_p^2 \leqslant C_p^2 I_D \leqslant C_p^2 (2/\pi) \|h\|_p \|l\|_q, \qquad \text{and} \qquad \|h\|_p \leqslant (2C_p^2/\pi)\|l\|_q,$$

Consider the case $b > 1$. Then the substitution of (1.15), (1.16) into (3.6) yield

$$\|h\|_p \leqslant C_p I_D^{\frac{p+1}{2p}} u_*^{-1/p} \leqslant u_*^{-1/p} C_p [(2/\pi)\|h\|_p \|l\|_q]^{\frac{p+1}{2p}},$$

and

$$\|h\|_p^{\frac{p-1}{2p}} \leqslant C_p u_*^{-1/p} [(2/\pi)\|l\|_q]^{\frac{p+1}{2p}}, \qquad \text{and} \qquad \|h\|_p \leqslant (C_p u_*^{-1/p})^{\frac{2p}{p-1}} (2/\pi)^{\frac{p+1}{p-1}} \|l\|_q]^{\frac{p+1}{p-1}},$$

and combining these two cases we have (1.3). Inequality $l_n \leqslant 2J_n$ (see (2.3)) yields the first estimate in (1.4). Relation (2.3) implies

$$\|J\|_p^p = \sum |J_n|^p \leqslant \sum (2/\pi)^{p/2} h_n^{p/2} l_n^{p/2} \leqslant (2/\pi)^{p/2} \|h\|_p^{p/2} \|l\|_p^{p/2},$$

and using (1.2) we obtain the second estimate in (1.4):

$$\|J\|_p \leqslant \sqrt{2/\pi} \|l\|_p^{1/2} [2\|l\|_p (1 + \alpha_p \|l\|_p^p)]^{1/2} = \frac{2}{\sqrt{\pi}} \|l\|_p (1 + \alpha_p \|l\|_p^p)^{1/2}.$$

Inequality $J_n^2 \leqslant 4h_n^2/\pi$ (see (2.3)) yields the first estimate in (1.5). Using (1.2) and $\|l\|_p \leqslant 2\|J\|_p$ (see (1.4)) we deduce that

$$\|h\|_p \leqslant 2\|l\|_p (1 + \alpha_p \|l\|_p^p) \leqslant 4\|J\|_p (1 + \alpha_p 2^p \|J\|_p^p). \quad \blacksquare$$

Consider now the examples, which proves Proposition 1.3, i.e., the exactness of some estimates.

Example 1. Define the sequence $h_n = N, |n| \leqslant N$, and $h_n = 0$, if $|n| > N$ and assume that $u_n = n, \quad n \in \mathbb{Z}$. Let $B_r = \{z : |z| < r\}, r > 0$. Introduce the function $g(k) = k + \frac{R^2}{k}, \ |k| > R = N\sqrt{8}$, which is the conformal mapping from $\mathbb{C} \setminus \overline{B_R}$ onto $\mathbb{C} \setminus [-2R, 2R]$.

Note that $\cup_{|n|\leqslant N}[-ih_n+u_n,u_n+ih_n]\subset\overline{B_R}$. Then Theorem 2.2 yields $\|h\|_\infty=N,\ \ u_*=1$, $\|h\|_2^2=2N^3,\ \ \|h\|_1=2N^2$ and using (1.15), (2.6) we obtain

$$N^2=\|h\|_\infty^2\leqslant 2Q_0=I_D\leqslant 16N^2.$$

Hence inequality (2.10) is precise. Moreover, estimate (2.2) yields $\frac{\pi N}{2}\leqslant\frac{\pi Q_0}{N}$ and $\|l\|_1\leqslant\frac{2\pi Q_0}{N}\leqslant 16\pi N$. Then we deduce that (1.2) at $p=1$, is precise. ■

In order to consider estimates (1.6) we need the following simple result.

Lemma 3.1. *Let $h\in l^\infty,\ u_1-u_0=u_0-u_{-1}=1$. Then*

$$l_0\leqslant((M^2-h_0^2+1)^2+4h_0^2)^{1/4},\qquad M=\min\{h_{-1},h_1\}.\tag{3.7}$$

Proof. Define the sequences

$$\widetilde{h}_m=\begin{cases}0, & \text{if }\ |m|\geqslant 2,\\ M, & \text{if }\ |m|=1,\\ h_n, & \text{if }\ m=0.\end{cases}\qquad\qquad \eta_m=\begin{cases}\widetilde{h}_m, & \text{if }\ \ m\neq 0,\\ 0, & \text{if }\ \ \ m=0.\end{cases}$$

Inequality (2.7) implies $\widetilde{l}_0\geqslant l_0$, then enough to show estimate (3.7) only for the sequence $h=\widetilde{h}$. We have

$$z(k,h)=\sqrt{(z(k,\eta))^2+|z(ih_0,\eta)|^2}.$$

Hence the maximum principle yields

$$l_0\leqslant\operatorname{Im}z(ih_0,\eta)\leqslant|\sqrt{M^2+(1+ih_0)^2}|\leqslant((M^2-h_0^2+1)^2+4h_0^2)^{1/4}.\qquad ■$$

Example 2. Introduce now the sequences

$$u_n=n,\quad n\in\mathbb{Z},\qquad h_n=\begin{cases}N-|n|, & \text{if }\ \ 0\leqslant|n|\leqslant N,\\ 0, & \text{if }\ \ |n|>N.\end{cases}$$

We estimate $\|l\|_2$. Using Lemma 3.1 we obtain for $|m|\leqslant N-2,\ \xi=N-|m|$:

$$l_m^4\leqslant((\xi-1)^2-\xi^2+1)^2+4\xi^2=4(\xi-1)^2+4\xi^2\leqslant 8N^2.$$

Moreover, the simple estimate $l_n\leqslant 2h_n$ implies $l_{N-1}\leqslant 1,\ l_{1-N}\leqslant 1$. Consider now $A=\sum_{|n|\leqslant N}l_n$. By the Theorem Ivanov-Pomerenke, $\frac{A}{4}$ is equal the capacity of the compact set $E=\cup_{|n|\leqslant N}[u_n-ih_n,u_n+ih_n]$. The capacity of the set E is less than the diameter which is equal to $2N$. Then $A\leqslant 8N$ and we have

$$\|l\|^2=\sum l_m^2\leqslant\sqrt[4]{8}\,\sqrt{N}\sum l_m\leqslant 8\sqrt[4]{8}\,N^{3/2}=BN^{3/2}.$$

Assume that the estimate $\|h\|\leqslant C\|l\|(1+\|l\|^p)$ holds for some constants $C,p>0$. Then for our Example 2 for large N we obtain $\|h\|\leqslant 2CB^{p/2}N^{3p/4}$. On the other hand we have $\|h\|\geqslant C_1N^{3/2}$ for some constant $C_1>0$. Then $N^{\frac{3}{2}(1-p/2)}\leqslant\frac{2C}{C_1}B^{p/2}$. It is possible only for

$p \geqslant 2$. Hence estimate $\|h\| \leqslant C\|l\|(1+\|l\|^p)$ is true only for some $p \geqslant 2$ (recall that $p=3$ in (1.6)). ■

We have considered the estimates for the weight $\omega_n \geqslant 1$. Now we obtain the counterexample, which shows impossibility of double-sided estimates in the space ℓ^∞_ω with $\omega_n \leqslant 1$.

Counterexample 3. Consider now the uniform comb with $u_n = \pi n$, $H = h_n = \|h\|_\infty$, $n \in \mathbb{Z}$. It is clear (see [LS]), that in this case $l = l_n = 2 \arcsin \operatorname{th} H$, $n \in \mathbb{Z}$. Then $l \to \pi$ as $H \to \infty$ and in this case for any sequence $\omega = (\omega_n)_{n\in\mathbb{Z}}$, $\omega_n > 0$, $n \in \mathbb{Z}$; $\sum\limits_{n\in\mathbb{Z}} \omega_n < +\infty$ and any $1 \leqslant p < +\infty$ the sequence h belongs to ℓ^p_ω and

$$\|h\|^p_{p,\omega} = \sum_{n\in\mathbb{Z}} h_n^p \omega_n = H^p \sum_{n\in\mathbb{Z}} \omega_n; \qquad \text{and} \qquad \|l\|^p_{p,\omega} = l^p \sum_{n\in\mathbb{Z}} \omega_n.$$

Hence the estimate $\|h\|_{p,\omega} \leqslant F(\|l\|_{p,\omega})$ is impossible for some function F. Moreover, by the same reason, the following estimate $\|h\|_\infty \leqslant F(\|l\|_\infty)$ is impossible too. ■

Proof of Theorem 1.5. Recall that for $h \in \ell^\infty, h_n \to 0$ as $n \to \infty$, we define the new sequence $\widetilde{h} = \widetilde{h}(h)$ by:

we take $n_1 \in \mathbb{Z}$ such that $\widetilde{h}_{n_1} = h_{n_1} = \max_{n\in\mathbb{Z}} h_n > 0$; assume that the numbers $n_1, n_2, \ldots, n_k$ have been defined, then we take n_{k+1} such that

$$\widetilde{h}_{n_{k+1}} = h_{n_{k+1}} = \max_{n\in B} h_n > 0, \quad B = \{n \in \mathbb{Z} : |u_n - u_{n_l}| > h_{n_l}, 1 \leqslant l \leqslant k\}.$$

Moreover, we let $\widetilde{h}_n = 0$, if the number $n \notin \{n_k, k \in \mathbb{Z}\}$. The Lindelöf principal yields $Q_0(\widetilde{h}) \leqslant Q_0(h)$. On the other hand open squares $P_k = (u_{n_k} - t_k, u_{n_k} + t_k) \times (-t_k, t_k), t_k \equiv h_{n_k} = \widetilde{h}_{n_k}, k \in Z$, does not overlap. Then applying (2.9) to the function $(z(k, \widetilde{h}) - k)$ and P_k, we obtain

$$2t_k^2 \leqslant \pi \int\int_{P_k} |z'(k,\widetilde{h}) - 1|^2 \, dudv, \tag{3.8}$$

and

$$Q_0(\widetilde{h}) = \frac{1}{2} I_D(\widetilde{h}) \geqslant \frac{1}{\pi^2} \sum_{k\geqslant 1} t_k^2 = \frac{1}{\pi^2} \|\widetilde{h}\|_2^2$$

which yields the first estimate in (1.20).

Let $\Omega_k = \{n \in \mathbb{Z} : u_n \in [u_{n_k} - t_k, u_{n_k} + t_k]\}$. By the Lindelöf principal, the gap length l_{n_k} such that $[u_n, u_n + ih_n], n \in \Omega_k$ increases if we take off all another slits. By the Theorem of Ivanov-Pomerenke (see Section 1), the sum of new gap lengths equals to $4 \times$ *capacity* of the set $E = \cup_{n\in\Omega_k} [u_n - h_n, u_n + h_n]$, which is less than the diameter of the set E. Then $\sum_{n\in\Omega_k} l_n(h) \leqslant 2\sqrt{2} t_k$, and using the last estimate we obtain

$$\pi Q_0(h) \leqslant \sum_{n\in\mathbb{Z}} h_n l_n \leqslant \sum_{k\geqslant 1} \sum_{n\in\Omega_k} h_n l_n \leqslant \sum_{k\geqslant 1} t_k \sum_{n\in\Omega_k} l_n \leqslant 2\sqrt{2} \sum_{k\geqslant 1} t_k^2 = 2\sqrt{2} \|\widetilde{h}\|_2^2, \tag{3.9}$$

since $h_n \leqslant t_k, n \in \Omega_k$ and the diameter of the set E is less than or equals $2\sqrt{2} t_k$. ■

Note that the proved Theorem shows that estimates (1.6),(2.11) are fulfilled for the weaker conditions on the sequence $u_n, n \in \mathbb{Z}$.

Recall the following identity for $v(z) = \operatorname{Im} k(z), z = x + iy$ from [KK1]:

$$v(x) = v_n(x)(1+V_n(x)), \quad V_n(x) = \frac{1}{\pi}\int\limits_{\mathbb{R}\setminus\gamma_n} \frac{v(t)dt}{|t-x|\, v_n(t)}, \quad v_n(x) = |(x-z_n^+)(x-z_n^-)|^{\frac{1}{2}}, \tag{3.10}$$

for all $x \in \gamma_n = (z_n^-, z_n^+)$. In order to prove Theorem 1.2 we need the following results.

Lemma 3.2. *Let $h \in \ell^\infty$ and $u_* > 0$ and $\xi = e^{\frac{\|h\|_\infty}{u_*}}$. Then the following estimates hold:*

$$s \equiv s(h) \equiv \inf |\sigma_n(h)| \leqslant u_* \leqslant \frac{\pi s}{2} \max\Big\{e^2, \xi^{\frac{5\pi}{2}}\Big\}, \tag{3.11}$$

$$1 + \frac{2\|h\|_\infty}{s\pi} \leqslant \xi^9, \tag{3.12}$$

$$\max_{n\in\gamma_n} V_n(x) \leqslant \frac{2\|h\|_\infty}{\pi s}, \quad n \in \mathbb{Z}, \tag{3.13}$$

$$2h_n \leqslant l_n(1 + \max_{n\in\gamma_n} V_n(x)) \leqslant l_n(1 + \frac{2\|h\|_\infty}{\pi s}) \leqslant l_n \xi^9, \quad n \in \mathbb{Z}. \tag{3.14}$$

Proof. Introduce the domain

$$G = \{z \in \mathbb{C} : h_+ \geqslant \operatorname{Im}\, z > 0,\ \operatorname{Re}\, z \in (-\frac{u_*}{2}, \frac{u_*}{2})\} \cup \{\operatorname{Im}\, z > h_+\}, \quad h_+ = \|h\|_\infty.$$

Let g be the conformal mapping from G onto $\mathbb{C}_+$, such that $g(iy) \sim iy$ as $y \nearrow +\infty$ and let α, β be images of the points $\frac{u_*}{2}, \frac{u_*}{2} + ih_+$ respectively. Define the function $f = \operatorname{Im} g$. Fix any $n \in \mathbb{Z}$. Then the maximum principle yields

$$y(k) = \operatorname{Im} z(k,h) \geqslant f(k - p_n), \quad k \in G + p_n, \quad p_n = \frac{1}{2}(u_{n-1} + u_n).$$

Due to the fact that these positive functions equal zero on the interval $(p_n - \frac{u_*}{2}, p_n + \frac{u_*}{2})$, we obtain

$$\frac{\partial y}{\partial v}(x) = \frac{\partial x}{\partial u}(x) \geqslant \frac{\partial f}{\partial v}(x - p_n), \quad x \in (p_n - \frac{u_*}{2}, p_n + \frac{u_*}{2}).$$

Then

$$z(u_n) - z(u_{n-1}) \geqslant \int\limits_{-u_*/2}^{u_*/2} \frac{\partial f}{\partial v}(x)\, dx = 2\alpha > 0,$$

and the estimate $s \leqslant u_*$ (see [KK1]) implies $2\alpha \leqslant s \leqslant u_*$. Let $w : \mathbb{C}_+ \to G$ be the inverse function for g, which is defined uniquely and the Christoffel-Schwartz formula yields

$$w(z) = \int_0^z \sqrt{\frac{t^2 - \beta^2}{t^2 - \alpha^2}} dt, \qquad 0 < \alpha < \beta.$$

Then we have

$$\frac{u_*}{2}=\int_0^\alpha\sqrt{\frac{\beta^2-t^2}{\alpha^2-t^2}}dt,\quad h_+=\int_\alpha^\beta\sqrt{\frac{\beta^2-t^2}{t^2-\alpha^2}}dt. \tag{3.15}$$

The first integral in (3.15) has the simple double-sided estimates

$$\alpha=\int_0^\alpha dt\leqslant\frac{u_*}{2}\leqslant\int_0^\alpha\frac{\beta dt}{\sqrt{\alpha^2-t^2}}=\frac{\beta\pi}{2},$$

that is

$$2\alpha\leqslant s\leqslant u_*\leqslant\pi\beta. \tag{3.16}$$

Consider the second integral in (3.15). Let $\varepsilon=\beta/\alpha\geqslant 5$ and using the new variable $t=\alpha\cosh r,\cosh\delta=\varepsilon$, we obtain

$$h_+=\alpha\int_0^\delta\sqrt{\varepsilon^2-\cosh^2 r}dr\geqslant\alpha\varepsilon\int_0^{\delta/2}\sqrt{1-\frac{\cosh^2 r}{\varepsilon^2}}dr\geqslant\beta\delta\frac{2}{5},$$

since for $r\leqslant\delta/2$ we have the simple inequality

$$\frac{\cosh^2 r}{\cosh^2\delta}\leqslant e^{-\delta}(1+e^{-\delta})^2\leqslant\varepsilon^{-1}(1+\varepsilon^{-1})^2.$$

Due to $\varepsilon\leqslant e^\delta$ we get $\varepsilon\leqslant\exp(5h_+/2\beta)$ and estimate (3.16) implies

$$\frac{1}{s}\leqslant\frac{\pi}{2u_*}\exp(\frac{5\pi}{2u_*}h_+),\quad if\quad\varepsilon\geqslant 5. \tag{3.17}$$

If $\varepsilon\leqslant 5$, then using (3.16) again we obtain

$$\frac{1}{s}\leqslant\frac{\varepsilon}{2\beta}\leqslant\frac{\pi\varepsilon}{2u_*}\leqslant\frac{\pi}{2u_*}5,\qquad if\quad\varepsilon\leqslant 5.$$

and the last estimate together with (3.17) yield (3.11),(3.12).

Identity (3.10) for $x\in\gamma_n=(z_n^-,z_n^+)$ implies

$$\pi V_n(x)=\int_{-\infty}^{z_n^--s}\frac{v(t)dt}{|t-x|v_n(t)}+\int_{z_n^++s}^\infty\frac{v(t)dt}{|t-x|v_n(t)}\leqslant\int_{-\infty}^{z_n^--s}\frac{h_+dt}{|t-z_n^-|^2}+\int_{z_n^++s}^\infty\frac{h_+dt}{|t-z_n^+|^2}\leqslant\frac{2h_+}{s}.$$

Using (3.10), (3.12) and simple inequality $v_n(z_n)\leqslant l_n/2$ we have (3.14). ∎

We prove the two-sided estimates of $h_n,l_n,\mu_n^\pm,J_n$ in the weight spaces.

Proof of Theorem 1.2. The first estimate in (1.8) follows from $h_n\leqslant 2\pi|\mu_n^\pm|$ (see [KK1]). The second one in (1.8) follows from $\|h\|_\infty\leqslant\sqrt{I_D}=\|J\|\leqslant\|J\|_p\leqslant\|J\|_{p,\omega}$ since $\omega_n\geqslant 1$ for any $n\in\mathbb{Z}$. Moreover, substituting (1.17) into $\|h\|_\infty\leqslant\sqrt{I_D}$, using (1.2) and $\|f\|_p\leqslant\|f\|_{p,\omega}$ for any f, we obtain the last estimate in (1.8).

Introduce the function $\xi=\exp\frac{\|h\|_\infty}{u_*}$. The first estimate in (1.9) follows from (2.1). Due to (3.14) we get $2h_n\leqslant\xi^9 l_n$, which yields the second estimate in (1.9).

The first estimate in (1.10) follows from (2.3). Using (2.3), (3.14) we have $J_n^2 \leqslant 2l_nh_n/\pi \leqslant (\xi^9/\pi)l_n^2$, which gives the second estimate in (1.10).

The first estimate in (1.11) follows from (2.3), (2.1). Using (2.3), (3.14) we obtain $h_n^2 \leqslant \xi^9 l_n h_n/2 \leqslant (\pi\xi^9/2)J_n^2$, which yields the second inequality in (1.11).

Identity $2\mu_n^{\pm} = \pm l_n[1 + V_n(z_n^{\pm})]^2$ (see [KK1]) implies $2|\mu_n^{\pm}| \geqslant l_n$, which yields the first inequality in (1.12). Moreover, using (3.14) we obtain the estimate $2|\mu_n^{\pm}| \leqslant \xi^{18}l_n$, which gives the second one in (1.12). $\blacksquare$

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